CS 70 Discrete Mathematics and Probability Theory
Fall 2020 Rao Final

Remote Proctoring Instructions.

- On questions 1-14: You need only give the answers on the **"Final (Short Answers)" gradescope assignment** within the 3 hour time period of the exam. (No justification is required.)
- On questions 15-20, the answers should be written on one side of one page per question including all subparts. Therefore, you will need to scan **6 sheets** of paper to a separate **gradescope assignment called Final (PDF and long answers.)**. You are welcome to write directly on a copy of the exam for questions 15-20 if you wish. The solutions are short, so one page per question is a **hard limit.**
- **The short answer assignment contains the questions. For the long answer questions download the PDF from the Final (PDF and long answers) gradescope assignment.**
- Both gradescope assignments will be available at 3:00 PM and the PDF for the **entire exam including short answers** will be available on the "Final(PDF and long answers)" assignment.
- There will be no clarifications. If a problem part has an error, we will remove it from the exam.
- **You have 180 minutes which includes the time to fill out the answers in the Final(Short Answers) gradescope assignment and then an extra twenty minutes to scan your paper solutions to the Final (PDF for long answers) assignment.**
- **For long answers with boxes, the answers must be in the boxes for any credit.**
- For individual emergencies, email fa20@eecs70.org or please use the disruption form at: "https://bit.ly/70disrupt"

Advice.

- The questions vary in difficulty. In particular, some of the long answers at the end are quite accessible, and even those are in not necessarily in order of difficulty. Also points (pts) are indicated in each problem heading in the pdf. **So do really scan over the exam a bit.**
- The question statement is your friend. Reading it carefully is a tool to get you to your "rational place".
- You may consult only *three sheet of notes on both sides*. Apart from that, you may not look at books, notes, etc. Calculators, phones, computers, and other electronic devices are NOT permitted.
- **You may, without proof, use theorems and lemmas that were proven in the notes and/or in lecture, unless otherwise stated.**

Major Gradescope Issues. If there is a global issue and it is not affecting you, please continue. If you are experiencing difficulties with gradescope, you may check your email, we will post a global message on piazza and bypass email preferences to inform you of what to do.

In particular, if the short answer gradescope becomes widely problematic we will ask you to scan one page per question with your answers **so keep paper available, one page for each of 13 short answer questions in addition to the 6 pages for the long questions or** 19 pages in total.

Please do not email in this global crash as we will not be able to deal with individual issues, just continue with your exam and write your answers on paper; one question per page for short answers, and one part per page for long answers.

Some Latex Commands for Gradescope.

You can (if you choose) use latex. It is fairly easy and satisfying.

Surround an expression by “$$ ... $$” on gradescope and you will be in latex.

Examples: “$$ A+B*D $$” will give: $A + B * D$.

There are useful commands:

1. “$$ A^2 $$” yields A^2
2. $$ \frac{a}{b} $$ yields $\frac{a}{b}$.
3. “\max ” yields max.
4. “a \b ” yields $a \geq b$.
5. “$$ (qˆ {-1} \pmod{p})$$” yields $(q^{-1} \pmod{p})$.
6. “{n \choose k-1}” yields $\binom{n}{k-1}$.
7. Grouping with “{ }”: “$$6 ˆ{G * H}$$” yields 6^{G*H}.

1. **Pledge.**

   Berkeley Honor Code: As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others.

   In particular, I acknowlege that:

   - I alone am taking this exam. Other than with the instructor and GSI, I will not have any verbal, written, or electronic communication about the exam with anyone else while I am taking the exam or while others are taking the exam.
   - I will not have any other browsers open while taking the exam.
   - I will not refer to any books, notes, or online sources of information while taking the exam, other than what the instructor has allowed.
   - I will not take screenshots, photos, or otherwise make copies of exam questions to share with others.

   Signed:______________

2. Long Ago. Pts: 2/2/2/2/3/3/3

1. $A \vee \neg(B \wedge C) \equiv A \vee (\neg B \vee \neg C)$

   ◯ True ◯ False

2. $\neg \forall x, \exists y, Q(x,y) \equiv \exists x, \exists y, \neg Q(x,y)$

   ◯ True ◯ False

3. $P \implies Q$ is logically equivalent to $\neg Q \vee P$.

   ◯ True ◯ False

4. Consider a stable matching instance where S is the job optimal stable pairing. Consider a run of the job-propose matching algorithm, where one candidate c rejects a job j that they should not have in one step. That is, c recieves an offer from j and j' and chooses j' instead of j, when j was ahead of j' in their preference list.

   Let P be the resulting pairing.

   (a) For every instance, job j cannot do better in P than in S. (Better means get a partner who they prefer more.)

   ◯ True ◯ False

   (b) Every candidate other than c does as well or better in P than in S.

   ◯ True ◯ False

   (c) Every job other than j does as well or better in P than in S.

   ◯ True ◯ False

3. Graphs: Pts: 3/3/3/2

1. Consider a graph on n vertices with exactly one cycle and m edges. What is the number of connected components? (Hint: $m \leq n$.)

2. For a tree on n vertices, what is the expected number of connected components if each edge is deleted with probability $1/3$?

3. If we delete every edge with probability $1/2$ from an Eulerian graph on n vertices, what is the expected number of odd degree vertices in the remaining graph?

4. Every simple cycle is 2-colorable.

○ True ○ False

4. Mostly Modular.Pts: 2/2/2/2/3/3/3/3/3/3/3

1. $\forall$ nonzero $x, y \in \mathbb{N}$ $gcd(x, y \mod x) = gcd(x, y)$.

   ◯ True ◯ False

2. $\forall$ nonzero $x, y \in \mathbb{N}$, $gcd(x, x \mod y) = gcd(x, y)$.

   ◯ True ◯ False

3. Give an example of positive integers for a and n where

$$(1 \cdot 2 \cdots (n-1))a^{n-1} \neq (1 \cdot 2 \cdots (n-1)) \pmod{n}.$$

   a: ☐ n: ☐

4. Let $S = \{x : x \in \{1, \ldots, 34\}$ and $gcd(x, 35) = 1\}$?
   (a) What is the size of the set S?

   ☐

   (b) What is $a^{|S|+1} \pmod{35}$ for $a \in S$?

   ☐

   (c) What is $a^{|S|+1} \pmod{35}$ for $a \notin S$?

   ☐

5. What is $18^{-1} \pmod{13}$?

   ☐

6. If $x = 1 \pmod{13}$ and $x = 0 \pmod{18}$ then what is $x \pmod{234}$? (Note: $234 = 18 \times 13$)

   ☐

7. For primes, p and q, where $e = d^{-1} \pmod{(p-1)(q-1)}$?
   (a) What is $a^{ed} \pmod{q}$? (Answer cannot use e or d, but may use numbers, a,p or q.)

   ☐

   (b) Find an $x \leq pq$, where $p | (a^{ed} - x)$. (Answer is an expression that may use a, p, and q.)

   ☐

5. Polynomials.Pts: 3/3/3

1. Given a polynomial, $x^3 + a_2x^2 + a_1x + a_0$ modulo 7 with roots at 3, 1, and 6. What is a_0? (Notice that the coefficient of x^3 is 1.)

2. Working $\pmod 5$, find a polynomial modulo 5 of degree 2 that has roots at 0 and 3, and goes through point $(2,3)$

3. Consider that one encodes a message of n numbers $\pmod m$, by forming a degree $n-1$ polynomial using the numbers as coefficients, and sending $2n-1$ points. If each point is erased with probability $1/2$, what is the probability that the original message can be reconstructed? (Hint: each pattern of erasures is equally probable.)

6. Countability/Computability Pts: 2/2/2/2/2/2

1. For every pair of distinct rational numbers there is a rational number in between them.

   ◯ True ◯ False

2. The rational numbers are uncountable.

   ◯ True ◯ False

3. There is a program that takes a program P, an input x, and a number n and determines whether P run on input x ever writes to memory location n.

   ◯ True ◯ False

4. There is a program that takes a program P, an input x, and a number n and determines whether P run on input x ever writes to any memory location $i \geq n$.

   ◯ True ◯ False

5. A program "knows" a real number if it takes an integer n and outputs the nth bit of the real number. (Note: positive values of n signify to the left of the decimal point, and negative ones to the right.)

   (a) There is a program that knows π.

   ◯ True ◯ False

   (b) For every real number x, there is a program that knows x.

   ◯ True ◯ False

7. A little counting.Pts: 3/3/3/3

1. What is the number of ways to have k strictly positive numbers that add up to n?

2. What is the number of ways to produce a sequence of numbers $0 < x_1 < x_2 < \cdots < x_k < n$?

3. What is the number of ways to produce a sequence of numbers $0 \leq x_1 \leq x_2 \leq \cdots \leq x_k < n$?

4. What is the number of poker hands that have at least 1 ace? (Recall that a poker hand is 5 cards from a 52 card deck.)

8. Probability Pts: 3/3/3/2/2/2/2/4/2/2/2/2

1. Consider rolling two six sided fair dice.
   (a) What is the probability that exactly one die is 6?

   (b) What is the probability that the sum of the two dice is 6?

   (c) What is the probability that the sum is 6 given that at least one die is at least 3?

   (d) The event of rolling a 6 on the first die is independent of the event that the dice sum to 7.

   ○ True ○ False

   (e) The event of rolling at least one 6 is independent of the event that the dice sum to 7.

   ○ True ○ False

2. Flip a coin until you repeat either heads or tails 2020 times. We will derive the probability that the first coin is the same as the last coin in the entire sequence of flips.
   (a) If the process stops after 2020 tosses, what is the probability that the first and last coin are the same?

   (b) If the process stops after 2021 tosses, what it the probability that the first and last coin are the same?

   (c) What is the probability that the first coin is the same as the last coin in the entire sequence of flips?

3. Which of the following are always true.
   (a) $E[10X] = 10E[X]$

   ○ True ○ False

   (b) $E[X^2] = E[X]^2$

   ○ True ○ False

   (c) $E[(X-Y)^2] = E[X^2] + E[Y^2] - 2E[X]E[Y]$

   ○ True ○ False

   (d) $Var(X+Y) = Var(X) + Var(Y)$

   ○ True ○ False

9. Marbles: Pts: 4/4

Consider two bags of marbles, the "majority red" bag has 6 red marbles and 4 blue marbles, and the "majority blue" bag has 3 red marbles and 7 blue marble, and each bag is chosen with probability $1/2$.

1. If you draw a blue marble where each marble in the bag is equally likely, what is the probability that the bag is the "majority blue" bag.

2. What is the probability that the next marble is blue?

10. Variance, covariance, tail bounds.Pts: 3/3/3/3/3

1. If $E[X] = 4$, and $E[Y] = 5$, and $E[XY] = E[X]E[Y]$, what is $\text{cov}(X,Y)$?

2. A student earns one standard deviation above the mean on both exam 1 and exam 2. We define random variables X and Y as the score of a randomly chosen student on exam 1 and exam 2 respectively. If $Var(X) = 1$, $Var(Y) = 1$ and $Cov(X,Y) = .5$, how many standard deviations above the mean did the student get on the sum of her two scores? (Recall, $Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$.)

3. For a random variable, X, where $X \geq -1$, $E[X] = 5$, and $E[X^2] = 26$. Give an upper bound $\Pr[X \geq 6]$. (It should be tight with respect to the appropriate inequality.)

4. For a random variable, X, where $E[X] = 5$, and $E[X^2] = 26$, give an upper bound $\Pr[X \geq 9]$. (It should be tight with respect to the appropriate inequality.)

5. Let X be $E[X] = 10$ and $Var[X] = \sigma^2$. Let $Y = \frac{X_1 + \cdots + X_n}{n}$ where X_i are i.i.d samples of X, for what value of n is $Pr[|Y - E[X]| \geq 1.0] \leq .05$? (Provide a bound that is as tight as possible using Chebyshev's inequality.)

11. Continuous: warmup. Pts: 2/2/2/3/3

Consider a continuous random variable, X, with pdf $f(x)$. (Answers below are a number or possibly expressions that involve the random variable and $E[\cdot]$ or $Var[\cdot]$.)

1. What is $\int_{-\infty}^{\infty} f(x)dx$?

2. $\int_{-\infty}^{\infty} xf(x)dx$?

3. $\int_{-\infty}^{\infty} x^2 f(x)dx$?

4. Consider $Y = 2X$ where $X \sim Expo(\lambda)$, $Y \sim Expo(\lambda')$, what is λ'?

5. Recall that for choosing a uniform point in a unit square the pdf is $f(x,y) = 1$ in the unit square and zero elsewhere. What is the pdf for choosing a uniform point in a 2×2 square? (Answer need only state the pdf inside the 2×2 square as outside it is zero.)

12. Distributions: continuous and discrete. Pts: 3/3/3/3/3/3/1/3

1. Given $X, Y \sim Binomial(n, p)$ what is the variance of $X+Y$?

2. What is $E[\min(X,Y,Z)]$ where $X,Y,Z \sim Geometric(p)$?

3. What is $E[\min(X,Y,Z)]$ where $X,Y,Z \sim Expo(\lambda)$?

4. Let $Z \sim Expo(\lambda)$ and $Y = \lceil Z \rceil$ (where $\lceil x \rceil$ is the lowest integer of value at least x). Note that the variable $Y \sim Geometric(p)$. What is the value of p in terms of λ?

5. Let $Y \sim Expo(\lambda)$, what is the conditional probability density function of Y if $Y \in [i, i+1]$ for a natural number i in the range $[i, i+1]$?
[A] $\frac{\lambda e^{-\lambda(x)}}{e^{-\lambda i}}$ [B] $\frac{\lambda e^{-\lambda(x)}}{(1-e^{-\lambda})}$ [C] $\frac{\lambda e^{-\lambda(x-i)}}{(1-e^{-\lambda})}$ [D] $\frac{\lambda e^{-\lambda(x-i)}}{e^{-\lambda}(1-e^{-\lambda})}$ [E] $\frac{\lambda e^{-\lambda(x-i)}}{(1-e^{-\lambda})^i}$

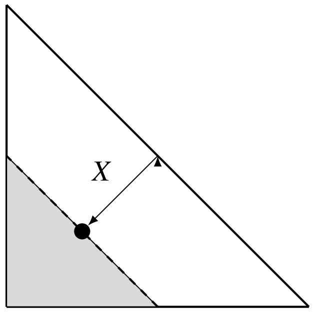
6. For $X \sim Geometric(p)$ and $Y \sim Poisson(X)$, what is $E[Y]$?

7. Consider a random variable $X = 2\ln Y$ where $Y \sim U[0,1]$.
   (a) What is the range of X? (The range is where the pdf of X is positive.)

   (b) What is the pdf of X on the range defined above? (Hint: $Pr[X \in [x, x+dx]] = Pr[Y \in [e^{x/2}, e^{(x+dx)/2}]$ and $e^{x+dx} \approx e^x(1+dx)$)

13. Continuous: Triangle. Pts: 2/3/3/2

Consider a right equilateral triangle of side lengths $1, 1$ and $\sqrt{2}$. Given a random point in the triangle, we define the random variable X as the distance from the hypotenuse as shown in the figure below.



1. What is the joint density function $f(x, y)$ for points inside the triangle? (Again, the point is chosen uniformly inside the entire triangle. Ignore the shading in the figure for now.)

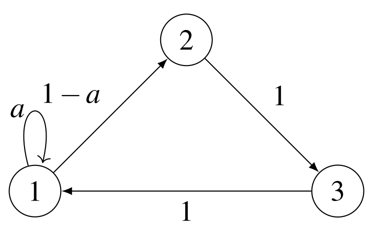
2. What is the area of the shaded triangle in terms of X? (Hint: range of X is $[0, \sqrt{2}/2]$)

3. What is the cdf of X for the range $x \in [0, \sqrt{2}/2]$?

4. What is the pdf of X for the range $x \in [0, \sqrt{2}/2]$?

14. Markov Chain. Pts: 2/2/2/2

Consider the following Markov chain.



1. For what value of a does the chain have a unique invariant distribution but does not always converge to it.

2. For $a = 1/2$, what is the stationary distribution?

$\pi(1)$: ______ $\pi(2)$: ______ $\pi(3)$: ______

Long Answers Starting From here.

15. Small faces. Pts: 2/2/4

Given a planar graph with minimum degree 3 with e edges, v vertices and f faces we will prove there is a face of length at most 5. (The length of a face is the number of edges along it.)

1. What is the sum of the face lengths, $\sum_{i=1}^{f} s_i$ where s_i is the size of face i, in terms of e?

2. Give a lower bound on e in terms of v. (Hint: the minimum degree is 3.)

3. Prove that there is a face of size at most 5. (Recall: Euler's formula $v + f = e + 2$.)

16. Balls and Bins.Pts: 3/3/2/2

Consider placing $5n$ balls into $3n$ bins uniformly at random. (Careful, the constants in front of the n's are important.)

1. What is the expected number of empty bins?

2. What is the variance of the number of empty bins?

3. What is the expected number of non-empty bins?

4. What is the variance of the number of non-empty bins (in terms of the answer for part (1),(2) and/or (3).)

17. Sequential Dice.Pts: 3/3/3

Consider rolling a dice repeatedly and until one gets two 6's in a row.

1. Draw a three state Markov chain where the states are labelled A,B, and C. Your chain should have a state C which is the "goal"; the previous two rolls were a 6. State A should indicate that one has not rolled any die or that the previous die is not 6.

2. What is the expected number of rolls to roll two 6's in a row?

3. What is the probability of rolling two 6's in a row prior to rolling a 5? (Hint: add a state to your previous Markov Chain and do a computation.)

18. Bayes Rule. Pts: 5

A doctor has information that 80% of the sick children in a neighborhood *have the flu* and the other 20% of sick children have measles. He further knows that the probability of a rash with measles, is 0.95, and that the probability of a rash with flu is .10. If a sick child has a rash, what is the probability the child has measles. (Show your work here. And use the box for your final answer.)

19. Close enough! Pts: 5

Given a circle (dartboard) of radius 1, choose two points at random on the dartboard uniformly, and let X and Y define the distance to the center. What is the probability that $|X-Y| \leq \delta$? (Recall that the pdf of both variables is $f(x) = 2x1\{x<1\}$)

20. Puzzler: Pts: 3/5

Consider the following game on an $n \times m$ grid, with two cooperating players. A key is hidden under a grid square and on each square there is a single coin that is either heads or tails. Player 1 knows the key location and *must flip exactly one coin.*

Player 2 should observe the pattern of heads and tails and produce the key location.

To reiterate, from an arbitrary initial setup of heads and tails on the grid, player 1 should flip exactly one coin to make a setup where player 2 can determine the location of the hidden key.

1. What is a strategy for the players to win on a 2×1 grid? (Hint: Think about $2x + y \pmod 2$ for $x, y \in \{0, 1\}$ and think of heads as 1 and 0 as tails.)

2. What is a strategy for the players to win on a $2^k \times 2^k$ grid? (Hint: use induction to find the column and row of the coin to flip. Notice, $2^k = (2 \times (2 \times 2^{k-1}).)$