CS 70 Discrete Mathematics and Probability Theory
Summer 2023 Huang, Suzani, and Tausik

HW 7

Due: Aug 9, 2023 11:59pm
Grace period until Aug 10, 2023 11:59pm

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Short Answer

Note 21

(a) Let X be uniform on the interval $[0,2]$, and define $Y = 4X^2 + 1$. Find the PDF, CDF, expectation, and variance of Y.

(b) Let X and Y have joint distribution

$$f(x,y) = \begin{cases} cxy + \frac{1}{4} & x \in [1,2] \text{ and } y \in [0,2] \\ 0 & \text{otherwise.} \end{cases}$$

Find the constant c. Are X and Y independent?

(c) Let $X \sim \text{Exp}(3)$.

(i) Find probability that $X \in [0,\, 1]$.

(ii) Let $Y = \lfloor X \rfloor$. For each $k \in \mathbb{N}$, what is the probability that $Y = k$? Write the distribution of Y in terms of one of the famous distributions; provide that distribution's name and parameters.

(d) Let $X_i \sim \text{Exp}(\lambda_i)$ for $i = 1, \dots, n$ be mutually independent. It is a (very nice) fact that $\min(X_1, \dots, X_n) \sim \text{Exp}(\mu)$. Find μ.

2 Uniform Estimation

Note 17
Note 21

Let $U_1, \dots, U_n \overset{\text{iid}}{\sim} \text{Uniform}(-\theta, \theta)$ for some unknown $\theta \in \mathbb{R}$, $\theta > 0$. We wish to estimate θ from the data $U_1, \dots, U_n$.

(a) Why would using the sample mean $\overline{U} = \dfrac{1}{n}\sum_{i=1}^{n} U_i$ fail in this situation?

(b) Find the PDF of U_i^2 for $i \in \{1, \ldots, n\}$.

(c) Consider the following variance estimate:

$$V = \frac{1}{n} \sum_{i=1}^{n} U_i^2.$$

Show that for large n, the distribution of V is close to one of the famous ones, and provide its name and parameters.

(d) Use part (c) to construct an unbiased estimator for θ^2 that uses all the data.

(e) Let $\sigma^2 = \text{Var}[U_i^2]$. We wish to construct a confidence interval for θ^2 with a significance level of δ, where $0 < \delta < 1$.

(i) Without any assumption on the magnitude of n, construct a confidence interval for θ^2 with a significance level of δ using your estimator from part (d).

(ii) Suppose n is large. Construct an approximate confidence interval for θ^2 with a significance level of δ using your estimator from part (d). You may leave your answer in terms of Φ and Φ^{-1}, the normal CDF and its inverse.

3 Darts with Friends

Michelle and Alex are playing darts. Being the better player, Michelle's aim follows a uniform distribution over a disk of radius 1 around the center. Alex's aim follows a uniform distribution over a disk of radius 2 around the center.

(a) Let the distance of Michelle's throw from the center be denoted by the random variable X and let the distance of Alex's throw from the center be denoted by the random variable Y.

- What's the cumulative distribution function of X?
- What's the cumulative distribution function of Y?
- What's the probability density function of X?
- What's the probability density function of Y?

(b) What's the probability that Michelle's throw is closer to the center than Alex's throw? What's the probability that Alex's throw is closer to the center?

(c) What's the cumulative distribution function of $U = \max\{X, Y\}$?

(d) What's the cumulative distribution function of $V = \min\{X, Y\}$?

(e) What is the expectation of the absolute difference between Michelle's and Alex's distances from the center, that is, what is $\mathbb{E}[|X - Y|]$? [*Hint*: Use parts (c) and (d), together with the continuous version of the tail sum formula, which states that $\mathbb{E}[Z] = \int_0^\infty \mathbb{P}[Z \geq z] dz$.]