CS 70 Summer 2023 — Discrete Mathematics and Probability Theory — Huang, Suzani, and Tausik — HW 5

Due: Jul 27, 2023 11:59pm
Grace period until Jul 28, 2023 11:59pm

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Pairs of Beads

Sinho has a set of $2n$ beads ($n \geq 2$) of n different colors, such that there are two beads of each color. He wants to give out pairs of beads as gifts to all the other $n-1$ TAs, and plans on keeping the final pair for himself (since he is, after all, also a TA). To do so, he first chooses two beads at random to give to the first TA he sees. Then he chooses two beads at random from those remaining to give to the second TA he sees. He continues giving each TA he sees two beads chosen at random from his remaining beads until he has seen all $n-1$ TAs, leaving him with just the two beads he plans to keep for himself. Prove that the probability that at least one of the other TAs (*not* including Sinho himself) gets two beads of the same color is at most $\frac{1}{2}$.

2 Pairwise Independence

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Recall that the events A_1, A_2, and A_3 are *pairwise independent* if for all $i \neq j$, A_i is independent of A_j. However, pairwise independence is a weaker statement than *mutual independence*, which requires the additional condition that $\mathbb{P}[A_1 \cap A_2 \cap A_3] = \mathbb{P}[A_1]\mathbb{P}[A_2]\mathbb{P}[A_3]$.

Suppose you roll two fair six-sided dice. Let A_1 be the event that the first die lands on 1, let A_2 be the event that the second die lands on 6, and let A_3 be the event that the two dice sum to 7.

(a) Compute $\mathbb{P}[A_1]$, $\mathbb{P}[A_2]$, and $\mathbb{P}[A_3]$.

(b) Are A_1 and A_2 independent?

(c) Are A_2 and A_3 independent?

(d) Are A_1, A_2, and A_3 pairwise independent?

(e) Are A_1, A_2, and A_3 mutually independent?

3 Cookie Jars

Note 15

You have two jars of cookies, each of which starts with n cookies initially. Every day, when you come home, you pick one of the two jars randomly (each jar is chosen with probability $1/2$) and eat one cookie from that jar. One day, you come home and reach inside one of the jars of cookies, but you find that is empty! Let X be the random variable representing the number of remaining cookies in non-empty jar at that time. What is the distribution of X?

4 Class Enrollment

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Note 19

Lydia has just started her CalCentral enrollment appointment. She needs to register for a geography class and a history class. There are no waitlists, and she can attempt to enroll once per day in either class or both. The CalCentral enrollment system is strange and picky, so the probability of enrolling successfully in the geography class on each attempt is p_g and the probability of enrolling successfully in the history class on each attempt is p_h. Also, these events are independent.

(a) Suppose Lydia begins by attempting to enroll in the geography class everyday and gets enrolled in it on day G. What is the distribution of G?

(b) Suppose she is not enrolled in the geography class after attempting each day for the first 7 days. What is $\mathbb{P}[G = i \mid G > 7]$, the conditional distribution of G given $G > 7$?

(c) Once she is enrolled in the geography class, she starts attempting to enroll in the history class from day $G+1$ and gets enrolled in it on day H. Find the expected number of days it takes Lydia to enroll in both the classes, i.e. $\mathbb{E}[H]$.

Suppose instead of attempting one by one, Lydia decides to attempt enrolling in both the classes from day 1. Let G be the number of days it takes to enroll in the geography class, and H be the number of days it takes to enroll in the history class.

(d) What is the distribution of G and H now? Are they independent?

(e) Let A denote the day she gets enrolled in her first class and let B denote the day she gets enrolled in both the classes. What is the distribution of A?

(f) What is the expected number of days it takes Lydia to enroll in both classes now, i.e. $\mathbb{E}[B]$?

(g) What is the expected number of classes she will be enrolled in by the end of 30 days?

5 Fishy Computations

Assume for each part that the random variable can be modelled by a Poisson distribution.

(a) Suppose that on average, a fisherman catches 20 salmon per week. What is the probability that he will catch exactly 7 salmon this week?

(b) Suppose that on average, you go to Fisherman's Wharf twice a year. What is the probability that you will go at most once in 2018?

(c) Suppose that in March, on average, there are 5.7 boats that sail in Laguna Beach per day. What is the probability there will be *at least* 3 boats sailing throughout the *next two days* in Laguna?

6 Dice Games

Note 20

(a) Alice and Bob are playing a game. Alice picks a random integer X between 0 and 100 inclusive, where each value is equally likely to be chosen. Bob then picks a random integer Y between 0 and X inclusive. What is $\mathbb{E}[Y]$?

(b) Alice rolls a die until she gets a 1. Let X be the number of total rolls she makes (including the last one), and let Y be the number of rolls on which she gets an even number. Compute $\mathbb{E}[Y \mid X = x]$, and use it to calculate $\mathbb{E}[Y]$.

(c) Bob plays a game in which he starts off with one die. At each time step, he rolls all the dice he has. Then, for each die, if it comes up as an odd number, he puts that die back, and adds a number of dice equal to the number displayed to his collection. (For example, if he rolls a one on the first time step, he puts that die back along with an extra die.) However, if it comes up as an even number, he removes that die from his collection.

What is the expected number of dice Bob will have after n time steps?