CS 70 Discrete Mathematics and Probability Theory
Summer 2023 Huang, Suzani, and Tausik

HW 4

Due: Jul 20, 2023 11:59pm
Grace period until Jul 21, 2023 11:59pm

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Counting, Counting, and More Counting

Note 10

The only way to learn counting is to practice, practice, practice, so here is your chance to do so. Although there are many subparts, each subpart is fairly short, so this problem should not take any longer than a normal CS70 homework problem. You do not need to show work, and **Leave your answers as an expression** (rather than trying to evaluate it to get a specific number).

(a) How many ways are there to arrange n 1s and k 0s into a sequence?

(b) How many 19-digit ternary (0,1,2) bitstrings are there such that no two adjacent digits are equal?

(c) A bridge hand is obtained by selecting 13 cards from a standard 52-card deck. The order of the cards in a bridge hand is irrelevant.

  i. How many different 13-card bridge hands are there?
  ii. How many different 13-card bridge hands are there that contain no aces?
  iii. How many different 13-card bridge hands are there that contain all four aces?
  iv. How many different 13-card bridge hands are there that contain exactly 4 spades?

(d) Two identical decks of 52 cards are mixed together, yielding a stack of 104 cards. How many different ways are there to order this stack of 104 cards?

(e) How many 99-bit strings are there that contain more ones than zeros?

(f) An anagram of ALABAMA is any re-ordering of the letters of ALABAMA, i.e., any string made up of the letters A, L, A, B, A, M, and A, in any order. The anagram does not have to be an English word.

i. How many different anagrams of ALABAMA are there?

ii. How many different anagrams of MONTANA are there?

(g) How many different anagrams of ABCDEF are there if:

i. C is the left neighbor of E

ii. C is on the left of E (and not necessarily E's neighbor)

(h) We have 8 balls, numbered 1 through 8, and 25 bins. How many different ways are there to distribute these 8 balls among the 25 bins? Assume the bins are distinguishable (e.g., numbered 1 through 25).

(i) How many different ways are there to throw 8 identical balls into 25 bins? Assume the bins are distinguishable (e.g., numbered 1 through 25).

(j) We throw 8 identical balls into 6 bins. How many different ways are there to distribute these 8 balls among the 6 bins such that no bin is empty? Assume the bins are distinguishable (e.g., numbered 1 through 6).

(k) There are exactly 20 students currently enrolled in a class. How many different ways are there to pair up the 20 students, so that each student is paired with one other student? Solve this in at least 2 different ways. **Your final answer must consist of two different expressions.**

(l) How many solutions does $x_0 + x_1 + \cdots + x_k = n$ have, if each x must be a non-negative integer?

(m) How many solutions does $x_0 + x_1 = n$ have, if each x must be a *strictly positive* integer?

(n) How many solutions does $x_0 + x_1 + \cdots + x_k = n$ have, if each x must be a *strictly positive* integer?

2 Shipping Crates

Note 10

A widget factory has four loading docks for storing crates of ready-to-ship widgets. Suppose the factory produces 8 indistinguishable crates of widgets and sends each crate to one of the four loading docks.

(a) How many ways are there to distribute the crates among the loading docks?

(b) Now, assume that any time a loading dock contains at least 5 crates, a truck picks up 5 crates from that dock and ships them away. (e.g., if 6 crates are sent to a loading dock, the truck removes 5, leaving 1 leftover crate still in the dock). We will now consider two configurations to be identical if, for every loading dock, the two configurations have the same number of leftover crates in that dock. How would your answer in the previous part compare to the number of outcomes given the new setup? Do not compute the actual value in this part, we will doing that in parts (c) - (e). We are looking for a qualitative answer (greater than part (a), equal to part (a), less than or equal to part (a), etc.) Justify your answer.

(c) We will now attempt to count the number of configurations of crates. First, we look at the case where crates are removed from the dock. How many ways are there to distribute the crates such that some crate gets removed from the dock?

(d) How many ways are there to distribute the crates such that no crates are removed from the dock; i.e. no dock receives at least 5 crates?

(e) Putting it together now, what are the total number of possible configurations for crates in the modified scenario? *Hint:* Observe that, regardless of which dock receives the 5 crates, we end up in the same situation. After all the shipping has been done, how many possible configurations of leftover crates in loading docks are there?

3 Fizzbuzz

Fizzbuzz is a classic software engineering interview question. You are given a natural number n, and for each integer i from 1 to n you have to print a line containing either:

- "fizzbuzz" if i is divisible by 15,
- "fizz" if i is divisible by 3 but not 15,
- "buzz" if i is divisible by 5 but not 15,
- or the integer itself if i is not divisible by 3 or 5.

Assume that n is a multiple of 15.

(a) How many printed lines will be "fizzbuzz"?

(b) How many printed lines will be "fizz"?

(c) How many printed lines will be "buzz"?

(d) How many printed lines will be an integer?

4 Is This CS 61A?

Note 10

Define a Scheme bracket sequence to be a sequence of **n** opening and **n** closing brackets. A bracket sequence is considered **valid** when each opening bracket has a corresponding closing bracket. That is, for $n = 3$, $(())()$ is a valid bracket sequence whereas $())(()$ is not. Notice that in a valid sequence, if you read from left to right, the number of opening brackets seen so far must always be greater than or equal to the number of closing brackets.

(a) Compute the total number of bracket sequences in terms of n. Don't worry about whether the bracket sequence is valid yet!

(b) Compute the total number of **invalid** bracket sequences in terms of n. (Hint: Suppose $s = s_1 s_2 \dots s_{2n}$ is an invalid bracket sequence. Consider the first prefix $s_1 s_2 \dots s_i$ where there are more closed brackets than open brackets. What happens if we flip (change each open bracket to a closed bracket and vice versa) the rest of the bracket sequence $s_{i+1} \dots s_{2n}$?)

(c) Compute the total number of valid bracket sequences in terms of n. You may find your answers to part (a) and part (b) to be helpful.

5 Proofs of the Combinatorial Variety

Note 10

Prove each of the following identities using a combinatorial proof.

(a) For every positive integer $n > 1$,

$$\sum_{k=0}^{n} k \cdot \binom{n}{k} = n \cdot \sum_{k=0}^{n-1} \binom{n-1}{k}.$$

(b) For each positive integer m and each positive integer $n > m$,

$$\sum_{a+b+c=m} \binom{n}{a} \cdot \binom{n}{b} \cdot \binom{n}{c} = \binom{3n}{m}.$$

(Notation: the sum on the left is taken over all triples of nonnegative integers (a,b,c) such that $a+b+c=m$.)

6 Is This EECS 126?

Note 13
Note 14

Youngmin loves birdwatching. There are three sites he birdwatches at: Site A, B, and C. He goes to Site A 60% of the time, Site B 35% of the time, and Site C 5% of the time. The probability of seeing a nightingale at each site is $\frac{1}{10}$, $\frac{3}{10}$, and $\frac{2}{5}$, respectively. Using the information above, answer the following questions.

(a) What is the total probability that Youngmin sees a nightingale?

(b) Given that Youngmin sees a nightingale, what is the probability that Youngmin went to site B?

(c) Given that Youngmin didn't go to site B, what is the probability that Youngmin doesn't see a nightingale?

7 Past Probabilified

Note 13

In this question we review some of the past CS70 topics, and look at them probabilistically.
For the following experiments,

i. Define an appropriate sample space Ω.

ii. Give the probability function $\mathbb{P}[\omega]$.

iii. Compute $\mathbb{P}[E_1]$.

iv. Compute $\mathbb{P}[E_2]$.

(a) Fix a prime $q > 2$, and uniformly sample twice with replacement from $\{0, \ldots, q-1\}$ (assume we have two $\{0, \ldots, q-1\}$-sided fair dice and we roll them). Then multiply these two numbers with each other modulo q.

Let $E_1 =$ The resulting product is 0.

Let $E_2 =$ The product is $(q-1)/2$.

(b) Make a graph on v vertices by sampling uniformly at random from all possible edges. Here, assume for each edge we flip a fair coin; if it comes up heads, we include the edge in the graph, and otherwise we exclude that edge from the graph.

Let $E_1 =$ The graph is complete.

Let $E_2 =$ vertex v_1 has degree d.

(c) Create a random stable matching instance by having each person's preference list be a uniformly random permutation of the opposite entity's list (make the preference list for each individual job and each individual candidate a random permutation of the opposite entity's list). Finally, create a uniformly random pairing by matching jobs and candidates up uniformly at random (note that in this pairing, (1) a candidate cannot be matched with two different jobs, and a job cannot be matched with two different candidates (2) the pairing does not have to be stable).

Let $E_1 =$ All jobs have distinct favorite candidates.

Let $E_2 =$ The resulting pairing is the candidate-optimal stable pairing.