CS 70   Discrete Mathematics and Probability Theory
Summer 2023 Huang, Suzani, and Tausik   **HW 2**

Due: Jul 6, 2023 11:59pm
Grace period until Jul 7, 2023 11:59pm

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Countability Practice

(a) Do $(0,1)$ and $\mathbb{R}_+ = (0,\infty)$ have the same cardinality? If so, either give an explicit bijection (and prove that it is a bijection) or provide an injection from $(0,1)$ to $(0,\infty)$ and an injection from $(0,\infty)$ to $(0,1)$ (so that by Cantor-Bernstein theorem the two sets will have the same cardinality). If not, then prove that they have different cardinalities.

(b) Is the set of strings over the English alphabet countable? (Note that the strings may be arbitrarily long, but each string has finite length. Also the strings need not be real English words.) If so, then provide a method for enumerating the strings. If not, then use a diagonalization argument to show that the set is uncountable.

(c) Consider the previous part, except now the strings are drawn from a countably infinite alphabet $\mathscr{A}$. Does your answer from before change? Make sure to justify your answer.

2 Counting Functions

Are the following sets countable or uncountable? Prove your claims.

(a) The set of all functions f from $\mathbb{N}$ to $\mathbb{N}$ such that f is non-decreasing. That is, $f(x) \leq f(y)$ whenever $x \leq y$.

(b) The set of all functions f from $\mathbb{N}$ to $\mathbb{N}$ such that f is non-increasing. That is, $f(x) \geq f(y)$ whenever $x \leq y$.

3 Countability and the Halting Problem

Using methods from countability, we will prove the Halting Problem is undecidable.

a) What is a reasonable representation for a computer program? Using this definition, show that the set of all programs are countable. *(Hint: Machine Code)*

b) The Halting Problem only considers programs which take a finite length input. Show that the set of all finite-length inputs is countable.

c) Assume that you have a program that tells you whether or not a given program halts on a specific input. Since the set of all programs and the set of all inputs are countable, we can enumerate them and construct the following table.

	x_1	x_2	x_3	x_4	...
p_1	H	L	H	L	...
p_2	L	L	L	H	...
p_3	H	L	H	L	...
p_4	L	H	L	L	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

An H (resp. L) in the ith row and jth column means that program p_i halts (resp. loops) on input x_j. Now write a program that is not within the set of programs in the table above.

d) Find a contradiction in part a and part c to show that the halting problem can't be solved.

4 Tenfold

(a) Suppose we have a program TenTimes which takes in two programs, F and G, some input y, as well as an integer z.

TenTimes(F, G, y, z) will run F and G - both with input y - at the same time, and start a timer as soon as they start running. If after exactly z seconds, F is at line ℓ and G is at line 10ℓ, TenTimes returns True. Otherwise, the program returns False. (If F or G halts before x seconds, then it stays on the line in which it halts.) Show that TenTimes is decidable.

(b) We now consider a program TenTimes2, which takes just the two programs, F and G, and some input y. If F and G are both run at the same time with input y, and at some point F is at line ℓ while G is simultaneously at line 10ℓ, TenTimes2 will return True. If this never occurs, TenTimes2 will return False. (Again, if F or G halts before x seconds, then it stays on the line in which it halts.) Show that TenTimes2 is undecidable.

5 Graph Basics

In the first few parts, you will be answering questions on the following graph G.

(a) What are the vertex and edge sets V and E for graph G?

(b) Which vertex has the highest in-degree? Which vertex has the lowest in-degree? Which vertices have the same in-degree and out-degree?
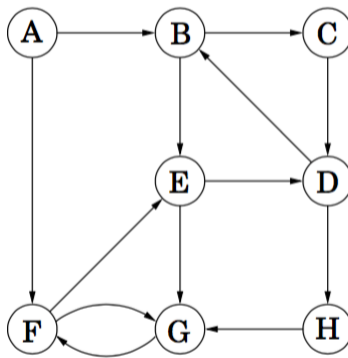
(c) What are the paths from vertex B to F, assuming no vertex is visited twice? Which one is the shortest path?

(d) Which of the following are cycles in G?

    i. $(B,C),(C,D),(D,B)$

    ii. $(F,G),(G,F)$

    iii. $(A,B),(B,C),(C,D),(D,B)$

    iv. $(B,C),(C,D),(D,H),(H,G),(G,F),(F,E),(E,D),(D,B)$

(e) Which of the following are walks in G?

    i. (E,G)

    ii. $(E,G),(G,F)$

    iii. $(F,G),(G,F)$

    iv. $(A,B),(B,C),(C,D),(H,G)$

    v. $(E,G),(G,F),(F,G),(G,C)$

    vi. $(E,D),(D,B),(B,E),(E,D),(D,H),(H,G),(G,F)$

(f) Which of the following are tours in G?

    i. (E,G)

    ii. $(E,G),(G,F)$

    iii. $(F,G),(G,F)$

    iv. $(E,D),(D,B),(B,E),(E,D),(D,H),(H,G),(G,F)$

    v. $(B,C),(C,D),(D,H),(H,G),(G,F),(F,E),(E,D),(D,B)$

In the following three parts, let's consider a general undirected graph G with n vertices $(n \geq 3)$. If true, provide a short proof. If false, show a counterexample.

(g) True/False: If each vertex of G has degree at most 1, then G does not have a cycle.

(h) True/False: If each vertex of G has degree at least 2, then G has a cycle.

(i) True/False: If each vertex of G has degree at most 2, then G is not connected.
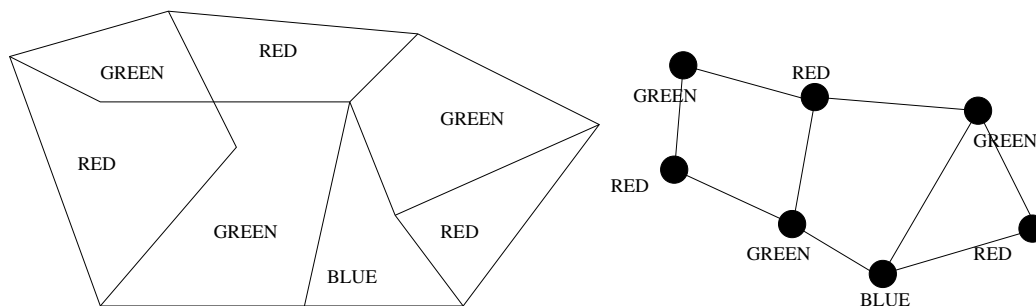
6 Coloring Countries

Figure 1: An island map that has been colored using three colors, and the corresponding graph.

Explorers have just discovered several new islands in the Pacific ocean! Each island is divided into several countries. As chief map-maker, your job is to make a map of each island, giving a color to each country so that *no two neighboring countries have the same color*. For example, the left side of Figure 1 shows a map that has been colored using three colors.

Unfortunately, you haven't been to the mapmaking store in a while, and so you only have six colors to work with: red, green, blue, purple, orange and almond toast. Fortunately, that's enough to color any map, as we shall see!

The right side of Figure 1 shows another way of looking at a map: make a vertex for each country, and draw an edge between two nodes if the countries are neighbors. The graph you get will be *planar*, meaning it can be laid out so that none of the edges cross each other.

(a) In order to color the map in Figure 1 so that no neighbors have the same color, you need at least three different colors. (To see why, try to color it using only red and green and see what happens.) Draw a map that needs at least *four* different colors, and then draw the corresponding planar graph.

(b) In order to see that six colors will be enough to color any map, prove the following theorem:

   Theorem: *Every planar graph can be colored with six colors, in such a way that no two neighboring vertices have the same color.*

   You may find the following lemma useful:

   Lemma: *Every planar graph (with at least one node) has a node with at most five neighbors. (We say the node has* degree *at most five.)*

   You don't need to prove the lemma, but you may assume it is true when proving the theorem.

   (In fact, it only ever takes four colors to color a planar graph – but that's much harder to prove. Search for *four color theorem*.)

7   Binary Trees

You may have seen the recursive definition of binary trees from previous classes. Here, we define binary trees in graph theoretic terms as follows (**Note:** here we will modify the definition of leaves slightly for consistency).

- A binary tree of height > 0 is a tree where exactly one vertex, called the **root**, has degree 2, and all other vertices have degrees 1 or 3. Each vertex of degree 1 is called a **leaf**. The **height** h is defined as the maximum length of the path between the root and any leaf.

- A binary tree of height 0 is the graph with a single vertex. The vertex is both a leaf and a root.

(a) Let T be a binary tree of height > 0, and let $h(T)$ denote it's height. Let r be the root in T and u and v be it's neighbors. Show that removing r from T will result in two binary trees, L, R with roots u and v respectively. Also, show that $h(T) = \max(h(L), h(R)) + 1$.

(b) Using the graph theoretic definition of binary trees, prove that the number of vertices in a binary tree of height h is at most $2^{h+1} - 1$.

(c) Prove that all binary trees with n leaves have $2n - 1$ vertices.

8   Does Euler Still Work?

In the country of Eulerville, we have k states, where each state can be modeled by a connected planar graph. Furthermore, all these states are connected through a series of non-intersecting highways that go from one state to another (in other words, the whole country can be thought of as a planar graph of planar graphs.)

One day, the leader of Eulerville decides to shutdown all highways that connect the states together, which results in the k states being disconnected from each other. Since Euler's formula, $v - e + f = 2$, requires the graph to be connected, how can we modify this formula to be correct for the now disconnected country of Eulerville? Be sure to prove your answer using induction. (Hint: we can still use Euler's formula for each of the k states, because each individual state is still planar and connected. Assume that after the highways are shutdown, we are left with V vertices, E edges, and F faces in the full planar graph)

Note 5

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