CS 70 Discrete Mathematics and Probability Theory
Summer 2023 Huang, Suzani, and Tausik

HW 1

Due: Jun 29, 2023 11:59pm
Grace period until Jun 30, 2023 11:59pm

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Proof by?

Note 2

(a) Prove that for any two integers x and y, if 10 does not divide xy, then 10 does not divide x and 10 does not divide y. In notation: $(\forall x, y \in \mathbb{Z})\ (10 \nmid xy) \implies ((10 \nmid x)\ \wedge\ (10 \nmid y))$. What proof technique did you use?

(b) Prove or disprove the contrapositive.

(c) Prove or disprove the converse.

2 Prove or Disprove

Note 2

For each of the following, either prove the statement, or disprove by finding a counterexample.

(a) $(\forall n \in \mathbb{N})$ if n is odd then $n^2 + 4n$ is odd.

(b) $(\forall a, b \in \mathbb{R})$ if $a + b \leq 15$ then $a \leq 11$ or $b \leq 4$.

(c) $(\forall r \in \mathbb{R})$ if r^2 is irrational, then r is irrational.

(d) $(\forall n \in \mathbb{Z}^+)\ 5n^3 > n!$. (Note: $\mathbb{Z}^+$ is the set of positive integers)

3 Proving Inequality

Note 3

For all positive integers $n \geq 1$, prove that

$$\frac{1}{3^1} + \frac{1}{3^2} + \ldots + \frac{1}{3^n} < \frac{1}{2}.$$

(Note: while you can use formula for an infinite geometric series to prove this, we would like you to use induction. If you're having trouble with the inductive step, try strengthening the inductive hypothesis. Can you prove an equality statement instead of an inequality?)

4 A Coin Game

Note 3

Your "friend" Stanley Ford suggests you play the following game with him. You each start with a single stack of n coins. On each of your turns, you select one of your stacks of coins (that has at least two coins) and split it into two stacks, each with at least one coin. Your score for that turn is the product of the sizes of the two resulting stacks (for example, if you split a stack of 5 coins into a stack of 3 coins and a stack of 2 coins, your score would be $3 \cdot 2 = 6$). You continue taking turns until all your stacks have only one coin in them. Stan then plays the same game with his stack of n coins, and whoever ends up with the largest total score over all their turns wins.

Prove that no matter how you choose to split the stacks, your total score will always be $\frac{n(n-1)}{2}$. (This means that you and Stan will end up with the same score no matter what happens, so the game is rather pointless.)

5 Nothing Can Be Better Than Something

Note 4

In the stable matching problem, suppose that some jobs and candidates have hard requirements and might not be able to just settle for anything. In other words, each job/candidate prefers being unmatched rather than be matched with those below a certain point in their preference list. Let the term "entity" refer to a candidate/job. A matching could ultimately have to be partial, i.e., some entities would and should remain unmatched.

Consequently, the notion of stability here should be adjusted a little bit to capture the autonomy of both jobs to unilaterally fire employees and/or employees to just walk away. A matching is stable if

- there is no matched entity who prefers being unmatched over being with their current partner;
- there is no matched/filled job and unmatched candidate that would both prefer to be matched with each other over their current status;
- there is no matched job and matched candidate that would both prefer to be matched with each other over their current partners; and
- similarly, there is no unmatched job and matched candidate that would both prefer to be matched with each other over their current status;
- there is no unmatched job and unmatched candidate that would both prefer to be with each other over being unmatched.

(a) Prove that a stable pairing still exists in the case where we allow unmatched entities.

(HINT: You can approach this by introducing imaginary/virtual entities that jobs/candidates "match" if they are unmatched. How should you adjust the preference lists of jobs/candidates, including those of the newly introduced imaginary ones for this to work?)

(b) As you saw in the lecture, we may have different stable matchings. But interestingly, if an entity remains unmatched in one stable matching, they must remain unmatched in any other stable matching as well. Prove this fact by contradiction.

6 Universal Preference

Note 4

Suppose that preferences in a stable matching instance are universal: all n jobs share the preferences $C_1 > C_2 > \cdots > C_n$ and all candidates share the preferences $J_1 > J_2 > \cdots > J_n$.

(a) What pairing do we get from running the algorithm with jobs proposing? Can you prove this happens for all n?

(b) What pairing do we get from running the algorithm with candidates proposing?

(c) What does this tell us about the number of stable pairings?

7 Preserving Set Operations

Note 0
Note 2

For a function f, define the image of a set X to be the set $f(X) = \{y \mid y = f(x) \text{ for some } x \in X\}$. Define the inverse image or preimage of a set Y to be the set $f^{-1}(Y) = \{x \mid f(x) \in Y\}$. Prove the following statements, in which A and B are sets.

Recall: For sets X and Y, $X = Y$ if and only if $X \subseteq Y$ and $Y \subseteq X$. To prove that $X \subseteq Y$, it is sufficient to show that $(\forall x)\ ((x \in X) \implies (x \in Y))$.

(a) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

(b) $f(A \cup B) = f(A) \cup f(B)$.