1 Joint Practice

Suppose that X and Y are random variables with joint density

$$f_{X,Y}(x,y) = \begin{cases} Ax^2y^2 & \text{if } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1, \\ 0 & \text{otherwise,} \end{cases}$$

where A is a positive constant.

(a) What is the value of A?

(b) What is the marginal density of X?

(c) What is $\text{cov}(X,Y)$?

2 Max of Uniforms

Let $X_1, \ldots X_n$ be independent $U[0,1]$ random variables, and let $X = \max(X_1, \ldots X_n)$. Compute each of the following in terms of n.

(a) What is the cdf of X?

(b) What is the pdf of X?

(c) What is $\mathbb{E}[X]$?

(d) What is $Var[X]$?

3 Exponential Expectation

(a) Let $X \sim \text{Exp}(\lambda)$. Use induction to show that $\mathbb{E}[X^k] = k!/\lambda^k$ for every $k \in \mathbb{N}$.

(b) For any $|t| < \lambda$, compute $\mathbb{E}[e^{tX}]$ directly from the definition of expectation.

(c) Using part (a), compute $\sum_{k=0}^{\infty} \frac{\mathbb{E}[X^k]}{k!} t^k$.

(d) Let $M(t) = \mathbb{E}[e^{tX}]$ be a function defined for all t such that $|t| < \lambda$. What is $\frac{dM(t)}{dt}\Big|_{t=0}$? What is $\frac{d^2M(t)}{dt^2}\Big|_{t=0}$? How does each of these relate to the mean and variance of an $\text{Exp}(\lambda)$ distribution?