1 Always, Sometimes, or Never

Note 5

In each part below, you are given some information about a graph G. Using only the information in the current part, say whether G will always be planar, always be non-planar, or could be either. If you think it is always planar or always non-planar, prove it. If you think it could be either, give a planar example and a non-planar example.

(a) G can be vertex-colored with 4 colors.

(b) G requires 7 colors to be vertex-colored.

(c) $e \leq 3v - 6$, where e is the number of edges of G and v is the number of vertices of G.

(d) G is connected, and each vertex in G has degree at most 2.

(e) Each vertex in G has degree at most 2.

2 Planarity

(a) Prove that $K_{3,3}$ is nonplanar.

(b) Consider graphs with the property T: For every three distinct vertices v_1, v_2, v_3 of graph G, there are at least two edges among them. Use a proof by contradiction to show that if G is a graph on ≥ 7 vertices, and G has property T, then G is nonplanar.

3 Hypercubes

Note 5

The vertex set of the n-dimensional hypercube $G = (V, E)$ is given by $V = \{0,1\}^n$ (recall that $\{0,1\}^n$ denotes the set of all n-bit strings). There is an edge between two vertices x and y if and only if x and y differ in exactly one bit position.

(a) Draw 1-, 2-, and 3-dimensional hypercubes and label the vertices using the corresponding bit strings.

(b) Show that the edges of an n-dimensional hypercube can be colored using n colors so that no pair of edges sharing a common vertex have the same color.

(c) Show that for any $n \geq 1$, the n-dimensional hypercube is bipartite.