1 Set Operations

Note 0

- $\mathbb{R}$, the set of real numbers
- $\mathbb{Q}$, the set of rational numbers: $\{a/b : a,b \in \mathbb{Z} \wedge b \neq 0\}$
- $\mathbb{Z}$, the set of integers: $\{\dots,-2,-1,0,1,2,\dots\}$
- $\mathbb{N}$, the set of natural numbers: $\{0,1,2,3,\dots\}$

(a) Given a set $A = \{1,2,3,4\}$, what is $\mathscr{P}(A)$ (Power Set)?

(b) Given a generic set B, how do you describe $\mathscr{P}(B)$ using set comprehension notation? (Set Comprehension is $\{x \mid x \in A\}$.)

(c) What is $\mathbb{R} \cap \mathscr{P}(A)$?

(d) What is $\mathbb{R} \cap \mathbb{Z}$?

(e) What is $\mathbb{N} \cup \mathbb{Q}$?

(f) What is $\mathbb{R} \setminus \mathbb{Q}$?

(g) If $S \subseteq T$, what is $S \setminus T$?

2 Preserving Set Operations

Note 0
Note 2

For a function f, define the image of a set X to be the set $f(X) = \{y \mid y = f(x) \text{ for some } x \in X\}$. Define the inverse image or preimage of a set Y to be the set $f^{-1}(Y) = \{x \mid f(x) \in Y\}$. Prove the following statements, in which A and B are sets. By doing so, you will show that inverse images preserve set operations, but images typically do not.

Recall: For sets X and Y, $X = Y$ if and only if $X \subseteq Y$ and $Y \subseteq X$. To prove that $X \subseteq Y$, it is sufficient to show that $(\forall x)\ ((x \in X) \implies (x \in Y))$.

(a) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

(b) $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$.

(c) $f(A \cap B) \subseteq f(A) \cap f(B)$, and give an example where equality does not hold.

(d) $f(A \setminus B) \supseteq f(A) \setminus f(B)$, and give an example where equality does not hold.

3 Inverses and Bijections

Note 0
Note 11

Recall that a function $f : A \to B$ is a bijection if it is an injection and a surjection, and it is invertible if there is a function $g : B \to A$ so that $g \circ f = \text{id}_A$ and $f \circ g = \text{id}_B$, where $\text{id}_A : A \to A$ and $\text{id}_B : B \to B$ are the identity functions.

(a) Prove that if $f : A \to B$ is invertible then it is a bijection.

(b) Prove that if $f : A \to B$ is a bijection then it is invertible.

(c) Let $g : B \to A$ be the inverse function for some bijection f. Is g necessarily a bijection?

4 Rationals and Irrationals

Note 2

Prove that the product of a non-zero rational number and an irrational number is irrational.